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Study on Some Properties of Fractional Analytic Function

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative, we study some properties of fractional analytic function, such as fractional Taylor's theorem, first fractional derivative test, and second fractional derivative test. The major methods used in this paper are fractional Rolle's theorem, fractional mean value theorem, product rule for fractional derivatives, and a new multiplication of fractional analytic functions. In fact, these new results are generalizations of those results in ordinary calculus.**

Keywords: **Jumarie's modified R-L fractional derivative, fractional analytic function, fractional Taylor's theorem, first and second fractional derivative test, fractional Rolle's theorem, fractional mean value theorem, product rule for fractional derivatives, new multiplication.**

I. INTRODUCTION

Fractional calculus studies the so-called fractional integral and derivative of real or complex order and their applications. It originated in 1695, in a letter written by L'Hospital to Leibniz, some problems are proposed, such as "what does fractional derivative mean?" or "what is the $1/2$ derivative of a function?". In the 18th and 19th centuries, many outstanding scientists focused their attention on this problem. For example, Euler, Laplace, Fourier, Abel, Liouville, Grünwald, Letnikov, Riemann, Laurent, or Heaviside. In the past few decades, fractional calculus has been applied to many fields, such as quantum mechanics, mathematical economics, viscoelasticity, dynamics, control theory, electronics [1-7]. However, the definition of fractional derivative is not unique, there are many useful definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modification of R-L fractional derivative [8-11]. Jumarie modified the definition of R-L fractional derivative with a new formula, and we obtained that the modified fractional derivative of a constant function is zero. Therefore, it is easier to connect fractional calculus with classical calculus by using this definition.

Based on Jumarie type of R-L fractional derivative, this paper studies some properties of fractional analytic function, such as fractional Taylor's theorem, first fractional derivative test, and second fractional derivative test. A new multiplication of fractional analytic function plays an important role in this article. The main methods used in this paper include fractional Rolle's theorem, fractional mean value theorem, and product rule for fractional derivatives. In fact, these results obtained by us are natural generalizations of those in traditional calculus.

II. PRELIMINARIES

First, we introduce the fractional calculus used in this paper and some important properties.

Definition 2.1 ([12])**:** Suppose that $0<\alpha\leq 1$, and x_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt, \quad (1)$$

where $\Gamma(w)=\int_0^{\infty} s^{w-1}e^{-s}ds$ is the gamma function defined on $w>0$. On the other hand, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, and it is called the n-th order α-fractional derivative of $f(x)$, where n is any positive integer.

Proposition 2.2 ([13])**:** *Suppose that* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{3}$$

Theorem 2.3 ([14])**:** *Let* $0<\alpha\leq 1$, *and* $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}a_k(x-x_0)^{k\alpha}$. *Then*

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{\left({}_{x_0}D_x^{\alpha}\right)^k[f_\alpha(x^\alpha)](x_0)}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{4}$$

Remark 2.4: In Theorem 2.3, $f_\alpha(x^\alpha)$ is called α-fractional analytic at $x=x_0$. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ $(a<b)$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Theorem 2.5 (fractional Rolle's theorem) ([15])**:** *Let* $0<\alpha\leq 1,(-1)^\alpha=-1$, *and* $a<b$. *If* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function on* $[a,b]$ *with* $f_\alpha(a^\alpha)=f_\alpha(b^\alpha)$, *then there exists* $\xi\in(a,b)$ *such that* $\left({}_{\xi}D_x^{\alpha}\right)[f_\alpha(x^\alpha)](\xi^\alpha)=0$.

Theorem 2.6 (fractional mean value theorem) ([15])**:** *Let* $0<\alpha\leq 1$, $(-1)^\alpha=-1$, *and* $a<b$. *If* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function on* $[a,b]$, *then there exists* $\xi\in(a,b)$ *such that* $f_\alpha(b^\alpha)-f_\alpha(a^\alpha)=\frac{\left({}_{\xi}D_x^{\alpha}\right)[f_\alpha(x^\alpha)](\xi^\alpha)}{\Gamma(\alpha+1)}(b-a)^\alpha$.

Definition 2.7 ([16])**:** If $0<\alpha\leq 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}.\end{aligned} \tag{8}$$

Theorem 2.8 (product rule for fractional derivatives) ([17])**:** *Let* $0<\alpha\leq 1$, c *be a real number, and* $f_\alpha(t^\alpha)$, $g_\alpha(t^\alpha)$ *be* α*-fractional analytic at* c. *Then*

$$\left({}_{c}D_t^{\alpha}\right)[f_\alpha(t^\alpha)\otimes g_\alpha(t^\alpha)]=\left({}_{c}D_t^{\alpha}\right)[f_\alpha(t^\alpha)]\otimes g_\alpha(t^\alpha)+f_\alpha(t^\alpha)\otimes\left({}_{c}D_t^{\alpha}\right)[g_\alpha(t^\alpha)]. \tag{9}$$

Theorem 2.9 ([18])**:** *If* $0<\alpha\leq 1$, *and* c *is a real number. Let* $f_\alpha(x^\alpha)$ *be a* α*-fractional analytic function defined on an interval containing* c, *then*

$$\left({}_{c}D_x^{\alpha}\right)[f_\alpha(x^\alpha)](c^\alpha)=\Gamma(\alpha+1)\cdot\lim_{x\to c}\frac{f_\alpha(x^\alpha)-f_\alpha(c^\alpha)}{(x-c)^\alpha}. \tag{10}$$

III. MAIN RESULTS

In this section, we provide some important theorems of fractional analytic function.

Theorem 3.1: *If* $0 < \alpha \leq 1$, $(-1)^\alpha = -1$, *and* $a < b$. *Let* $f_\alpha(x^\alpha)$ *be a* α*-fractional analytic function on* $[a, b]$.

Case1. If $\left({}_yD_x^\alpha\right)[f_\alpha(x^\alpha)](y) > 0$ *for all* $y \in (a, b)$, *then* $f_\alpha(x^\alpha)$ *is increasing on* $[a, b]$.

Case2. If $\left({}_yD_x^\alpha\right)[f_\alpha(x^\alpha)](y) < 0$ *for all* $y \in (a, b)$, *then* $f_\alpha(x^\alpha)$ *is decreasing on* $[a, b]$.

Case3. If $\left({}_yD_x^\alpha\right)[f_\alpha(x^\alpha)](y) = 0$ *for all* $y \in (a, b)$, *then* $f_\alpha(x^\alpha)$ *is constant on* $[a, b]$.

Proof Let $x_1 < x_2$ be any two points in (a, b). By fractional mean value theorem, we know that there exists a number c such that $x_1 < c < x_2$, and

$$f_\alpha(x_2^\alpha) - f_\alpha(x_1^\alpha) = \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(\alpha+1)}(x_2 - x_1)^\alpha. \tag{11}$$

Case 1. Since $\left({}_yD_x^\alpha\right)[f_\alpha(x^\alpha)](y) > 0$ for all $y \in (a, b)$ and $x_2 - x_1 > 0$, it follows that $f_\alpha(x_2^\alpha) - f_\alpha(x_1^\alpha) > 0$ which implies that $f_\alpha(x_1^\alpha) < f_\alpha(x_2^\alpha)$. Thus, $f_\alpha(x^\alpha)$ is increasing on $[a, b]$.

Case 2. Since $\left({}_yD_x^\alpha\right)[f_\alpha(x^\alpha)](y) < 0$ for all $y \in (a, b)$ and $x_2 - x_1 > 0$, it follows from fractional mean value theorem that $f_\alpha(x_1^\alpha) > f_\alpha(x_2^\alpha)$. So, $f_\alpha(x^\alpha)$ is decreasing on $[a, b]$.

Case 3. Since $\left({}_yD_x^\alpha\right)[f_\alpha(x^\alpha)](y) = 0$ for all $y \in (a, b)$ and $x_2 - x_1 > 0$, it follows that $f_\alpha(x_1^\alpha) = f_\alpha(x_2^\alpha)$. Hence, $f_\alpha(x^\alpha)$ is constant on $[a, b]$. Q.e.d.

Theorem 3.2 (fractional Taylor's theorem)**:** *Let* $0 < \alpha \leq 1$, $(-1)^\alpha = -1$, *n be a positive integer, c be a real number, and let* $f_\alpha(x^\alpha)$ *be a* α*-fractional analytic function in an interval I containing c. Then for each x in I, there exists* ξ *between x and c such that*

$$f_\alpha(x^\alpha) = f_\alpha(c^\alpha) + \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(\alpha+1)}(x-c)^\alpha + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \cdots + \frac{\left({}_cD_x^\alpha\right)^n[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(n\alpha+1)}(x-c)^{n\alpha} + R_{\alpha,n}(x^\alpha), \tag{12}$$

where $R_{\alpha,n}(x^\alpha) = \frac{\left({}_\xi D_x^\alpha\right)^{n+1}[f_\alpha(x^\alpha)](\xi^\alpha)}{\Gamma((n+1)\alpha+1)}(x-c)^{(n+1)\alpha}$.

Proof Since $R_{\alpha,n}(x^\alpha) = f_\alpha(x^\alpha) - P_{\alpha,n}(x^\alpha)$, where

$$P_{\alpha,n}(x^\alpha) = f_\alpha(c^\alpha) + \frac{\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(\alpha+1)}(x-c)^\alpha + \frac{\left({}_cD_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \cdots + \frac{\left({}_cD_x^\alpha\right)^n[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(n\alpha+1)}(x-c)^{n\alpha} \tag{13}$$

is the nth α-fractional Taylor polynomial for $f_\alpha(x^\alpha)$. Let $g_\alpha(t^\alpha)$ be a α-fractional analytic function defined by

$$g_\alpha(t^\alpha) = f_\alpha(x^\alpha) - f_\alpha(t^\alpha) - \frac{\left({}_tD_x^\alpha\right)[f_\alpha(x^\alpha)](t^\alpha)}{\Gamma(\alpha+1)} \otimes (x-t)^\alpha - \cdots - \frac{\left({}_tD_x^\alpha\right)^n[f_\alpha(x^\alpha)](t^\alpha)}{\Gamma(n\alpha+1)} \otimes (x-t)^{n\alpha} - R_{\alpha,n}(x^\alpha) \cdot \frac{(x-t)^{(n+1)\alpha}}{(x-c)^{(n+1)\alpha}}, \tag{14}$$

where t between x and c.

Thus, by product rule for fractional derivatives, we have

$$\begin{aligned}\left({}_cD_t^\alpha\right)[g_\alpha(t^\alpha)] &= -\left({}_tD_x^\alpha\right)[f_\alpha(x^\alpha)](t^\alpha) + \left({}_tD_x^\alpha\right)[f_\alpha(x^\alpha)](t^\alpha) - \frac{\left({}_tD_x^\alpha\right)^2[f_\alpha(x^\alpha)](t^\alpha)}{\Gamma(\alpha+1)}(x-t)^\alpha + \cdots \\ &\quad - \frac{\left({}_tD_x^\alpha\right)^{n+1}[f_\alpha(x^\alpha)](t^\alpha)}{\Gamma(n\alpha+1)}(x-t)^{n\alpha} + \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}R_{\alpha,n}(x^\alpha) \cdot \frac{(x-t)^{n\alpha}}{(x-c)^{(n+1)\alpha}} \\ &= -\frac{\left({}_tD_x^\alpha\right)^{n+1}[f_\alpha(x^\alpha)](t^\alpha)}{\Gamma(n\alpha+1)}(x-t)^{n\alpha} + \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}R_{\alpha,n}(x^\alpha) \cdot \frac{(x-t)^{n\alpha}}{(x-c)^{(n+1)\alpha}}. \end{aligned} \tag{15}$$

Furthermore, for a fixed x,

$$g_\alpha(c^\alpha) = f_\alpha(x^\alpha) - [P_{\alpha,n}(x^\alpha) + R_{\alpha,n}(x^\alpha)] = f_\alpha(x^\alpha) - f_\alpha(x^\alpha) = 0. \tag{16}$$

And

$$g_\alpha(x^\alpha) = f_\alpha(x^\alpha) - f_\alpha(x^\alpha) = 0. \tag{17}$$

Therefore, by fractional Rolle's theorem, there is a number ξ between x and c such that $\left({}_\xi D_t^\alpha\right)[g_\alpha(t^\alpha)](\xi^\alpha) = 0$. On the other hand, since

$$\left({}_\xi D_t^\alpha\right)[g_\alpha(t^\alpha)](\xi^\alpha) = -\frac{\left({}_\xi D_x^\alpha\right)^{n+1}[f_\alpha(x^\alpha)](\xi^\alpha)}{\Gamma(n\alpha+1)}(x-\xi)^{n\alpha} + \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} R_{\alpha,n}(x^\alpha) \cdot \frac{(x-\xi)^{n\alpha}}{(x-c)^{(n+1)\alpha}} = 0\,. \tag{18}$$

It follows that

$$R_{\alpha,n}(x^\alpha) = \frac{\left({}_\xi D_x^\alpha\right)^{n+1}[f_\alpha(x^\alpha)](\xi^\alpha)}{\Gamma((n+1)\alpha+1)}(x-c)^{(n+1)\alpha}\,.$$

Finally, since $g_\alpha(c^\alpha) = 0$, we have

$$f_\alpha(x^\alpha) = f_\alpha(c^\alpha) + \frac{\left({}_c D_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(\alpha+1)}(x-c)^\alpha + \frac{\left({}_c D_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \cdots + \frac{\left({}_c D_x^\alpha\right)^n[f_\alpha(x^\alpha)](c^\alpha)}{\Gamma(n\alpha+1)}(x-c)^{n\alpha}$$

$$+R_{\alpha,n}(x^\alpha).$$

Q.e.d.

Theorem 3.3 (first fractional derivative test)**:** *If* $0 < \alpha \le 1$, $(-1)^\alpha = -1$, *and* $a < c < b$. *Let* $f_\alpha(x^\alpha)$ *be a* α*-fractional analytic function on* $[a,b]$.

Case1. If $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]$ *changes from negative to positive at c, then* $f_\alpha(c^\alpha)$ *is a relative minimum.*

Case2. If $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]$ *changes from positive to negative at c, then* $f_\alpha(c^\alpha)$ *is a relative maximum.*

Case3. If $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]$ *is positive on both sides of c or negative on both sides of c, then* $f_\alpha(c^\alpha)$ *is neither a relative minimum nor a relative maximum.*

Proof *Case1.* If $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]$ changes from negative to positive at c, then $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)] < 0$ for all $x \in (a,c)$ and $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)] > 0$ for all $x \in (c,b)$. By Theorem 3.1, $f_\alpha(x^\alpha)$ is decreasing on $[a,c]$ and increasing on $[c,b]$. Thus, $f_\alpha(c^\alpha)$ is a relative minimum. *Case 2* can be proved in a similar way.

Case3. If $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]$ is positive on both sides of c or negative on both sides of c. Then $f_\alpha(x^\alpha)$ is increasing on both sides of c or decreasing on both sides of c, and hence $f_\alpha(c^\alpha)$ is neither a relative minimum nor a relative maximum.

Q.e.d.

Theorem 3.4 (second fractional derivative test)**:** *Let* $0 < \alpha \le 1$, $(-1)^\alpha = -1$, *and* $a < c < b$. *Let* $f_\alpha(x^\alpha)$ *be a* α*-fractional analytic function on* $[a,b]$ *and* $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha) = 0$.

Case1. If $\left({}_a D_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha) > 0$, *then* $f_\alpha(c^\alpha)$ *is a relative minimum.*

Case2. If $\left({}_a D_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha) < 0$, *then* $f_\alpha(c^\alpha)$ *is a relative maximum.*

Proof *Case 1*. If $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha) = 0$ and $\left({}_a D_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha) > 0$. Then by Theorem 2.9, we have

$$\left({}_a D_x^\alpha\right)^2[f_\alpha(x^\alpha)](c^\alpha) = \Gamma(\alpha+1) \cdot \lim_{x\to c} \frac{\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)] - \left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)](c^\alpha)}{(x-c)^\alpha} = \Gamma(\alpha+1) \cdot \lim_{x\to c} \frac{\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]}{(x-c)^\alpha} > 0. \tag{19}$$

If $x < c$, then $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)] < 0$. If $x > c$, then $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)] > 0$. Hence, $\left({}_a D_x^\alpha\right)[f_\alpha(x^\alpha)]$ changes from negative to positive at c, and the first fractional derivative test implies that $f_\alpha(c^\alpha)$ is a relative minimum. *Case 2* can be proved in a similar way. Q.e.d.

IV. CONCLUSION

This paper studies some properties of fractional analytic function, such as fractional Taylor's theorem, first fractional derivative test, and second fractional derivative test. Based on Jumarie type of R-L fractional derivative, the main methods used in this paper are fractional Rolle's theorem, fractional mean value theorem, product rule for fractional derivatives, and a new multiplication of fractional analytic functions. In fact, these new results are generalizations of ordinary calculus results. In the future, we will continue to use these important theorems to solve the problems in engineering mathematics and fractional calculus.

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